The Einstein-scalar field constraint system in the positive case.

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Abstract

We prove the existence of solutions to the conformal Einstein-scalar constraint system of equations for closed compact Riemannian manifolds in the positive case. Our results apply to the vacuum case with positive cosmological constant and to the massive Klein-Gordon setting.

1 Introduction

1.1 The Einstein constraint equations in a scalar-field theory and the conformal method.

The constraint equations arise in general relativity. A space-time is a Lorentzian manifold $(\tilde{M}, \tilde{g})$ of dimension $n+1$ that solves the Einstein equations:

$$\text{Ric}_{ij}(\tilde{g}) - \frac{1}{2}R(\tilde{g})\tilde{g}_{ij} = 8\pi T_{ij} \tag{1.1}$$

where $R(\tilde{g})$ and $\text{Ric}(\tilde{g})$ are respectively the scalar curvature and the Ricci curvature of $\tilde{g}$ and T_{ij} is the stress-energy tensor. In a scalar-field theory the expression of T involves a scalar field $\psi \in C^\infty(\tilde{M})$, a potential $V \in C^\infty(\mathbb{R})$, and the metric $\tilde{g}$, and is written as

$$T_{ij} = \nabla_i \psi \nabla_j \psi - \left(\frac{1}{2}|\nabla\psi|^2_{\tilde{g}} + V(\psi)\right)\tilde{g}_{ij}.$$

This setting covers several usual physical cases: for instance $V = 0, \psi = 0$ yields the vacuum constraint equations, $V \equiv \Lambda$ and $\psi = 0$ yields the vacuum case with a cosmological constant, and $V(\psi) = \frac{1}{2}m\psi^2$ yields the massive Klein-Gordon setting. An initial data set for the Einstein equations consists of $(M, g, K, \psi, \tilde{\pi})$, where (M, g) is a n-dimensional closed compact Riemannian manifold ($n \geq 3$), k is a symmetric $(2, 0)$-tensor, and ψ and $\tilde{\pi}$ are smooth functions in M. The Cauchy problem in general relativity deals with constructing space-time developments for given initial data sets $(M, g, K, \psi, \tilde{\pi})$. Such a development consists of a Lorentzian manifold $(M \times \mathbb{R}, \tilde{g})$ and of a smooth function $\tilde{\psi}$ on $M \times \mathbb{R}$ such that $(M \times \mathbb{R}, \tilde{g})$ solves the Einstein equations (1.1), g is the Riemannian metric induced by $\tilde{g}$ on M, K is the second fundamental form of the embedding $M \subset M \times \mathbb{R}$ and ψ and $\tilde{\pi}$ are respectively the scalar field and its temporal derivative on M, that is $\tilde{\psi}_{|M} = \psi$ and $\partial_t \tilde{\psi}_{|M} = \tilde{\pi}$. A necessary condition for the existence of a space-time development of an initial data set $(M, g, K, \psi, \tilde{\pi})$ is found applying the Gauss and Codazzi equations to (1.1)

and yields the following well-known system of equations:

$$\begin{cases} R(g) + \text{tr}_g K^2 - ||K||_g^2 = \tilde{\pi}^2 + |\nabla\psi|_g^2 + 2V(\psi) \ , \\ \partial_i(\text{tr}_g K) - K_{i,j}^j = \tilde{\pi}\partial_i\psi \ , \end{cases} \tag{1.2}$$

where $R(g)$ is the scalar curvature of (M, g). As shown first by Choquet-Bruhat [10] for the vacuum case ($\psi = \tilde{\pi} = 0$) and later by Choquet-Bruhat-Isenberg-Pollack in the general case [5], the system (1.2) is also a sufficient condition on $(M, g, K, \psi, \tilde{\pi})$ for the existence of a space-time development. This development is unique as shown by Choquet-Bruhat and Geroch [4]. A survey reference on the subject is Chruściel-Galloway-Pollack [7].

Solving (1.2) provides admissible initial data for the Einstein equations. A method that turned out to be very effective to solve (1.2) is the conformal method, initiated by Lichnerowicz [15]. It consists in turning (1.2) into a determined system by specifying some initial "free" data and to solve the system in the remaining "determined" initial data. The set of free data consists of (ψ, τ, π, U) where ψ, τ, π are smooth functions in M and U is a smooth symmetric traceless and divergence-free $(2, 0)$-tensor in M. Given (ψ, τ, π, U) an initial free data set, the conformal method yields a system of two equations, often referred to as the conformal constraint system of equations, whose unknowns are a smooth positive function φ in M and a smooth vector field W in M. The conformal constraint system is written as

$$\begin{cases} \triangle_g\varphi + \mathcal{R}_\psi\varphi = \mathcal{B}_{\tau,\psi,V}\varphi^{2^*-1} + \dfrac{\mathcal{A}_{\pi,U}(W)}{\varphi^{2^*+1}} \ , & (1.3) \\[2mm] \triangle_{g,conf}W = \dfrac{n-1}{n}\varphi^{2^*}\nabla\tau - \pi\nabla\psi \ , & (1.4) \end{cases}$$

where we have let:

$$\begin{aligned} \mathcal{R}_\psi &= c_n\left(R(g) - |\nabla\psi|_g^2\right), \\ \mathcal{B}_{\tau,\psi,V} &= c_n\left(2V(\psi) - \frac{n-1}{n}\tau^2\right), \\ \mathcal{A}_{\pi,U}(W) &= c_n\left(|U + \mathcal{L}_g W|_g^2 + \pi^2\right), \end{aligned} \tag{1.5}$$

and $c_n = \frac{n-2}{4(n-1)}$. The notation $\mathcal{A}_{\pi,U}(W)$ emphasizes the dependency with respect to W, which is given by the second equation. In (1.5) we adopt similar notations to those in Choquet-Bruhat, Isenberg and Pollack [6] except for the minus sign on $\mathcal{B}_{\tau,\psi,V}$. Also, in (1.3)-(1.4), $\triangle_g = -\text{div}_g\nabla$ is the Laplace-Beltrami operator, with nonnegative eigenvalues, $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, $\triangle_{g,conf}W = \text{div}_g(\mathcal{L}_g W)$ and $\mathcal{L}_g W$ is the symmetric trace-free part of ∇W:

$$\mathcal{L}_g W_{ij} = W_{i,j} + W_{j,i} - \frac{2}{n}\text{div}_g W g_{ij}. \tag{1.6}$$

The first equation is referred to as the Einstein-Lichnerowicz equation while the second one is referred to as the momentum constraint. Smooth vector fields in the kernel of $\mathcal{L}_g$ are called conformal Killing vector fields. Since M is compact without boundary, the integration by parts formula gives, for any smooth vector field W:

$$\triangle_{g,conf}W = 0 \iff \mathcal{L}_g W = 0.$$

Given an initial data set (ψ, τ, π, U), if (φ, W) solves (1.3)-(1.4) then

$$\left(M, \varphi^{\frac{4}{n-2}}g, \frac{\tau}{n}\varphi^{\frac{4}{n-2}}g + \varphi^{-2}(U + \mathcal{L}_g W), \psi, \varphi^{-\frac{2n}{n-2}}\pi\right) \tag{1.7}$$

is a solution of (1.2). In this case τ is the mean curvature of $(M, \varphi^{\frac{4}{n-2}} g)$ embedded in its space-time development, ψ is the scalar-field restricted to M and, up to a conformal factor, π is the time derivative of the scalar-field in M. We refer to Choquet-Bruhat, Isenberg and Pollack [6] and Bartnik-Isenberg [2] for more developments on the conformal method. Solving the constraint system in usual cases such as the massive Klein-Gordon setting or the positive cosmological constant case amounts to solve (1.3)-(1.4) for a good choice of the initial data.

1.2 Statement of the results

In this paper we focus on the conformal constraint system (1.3)-(1.4) in the case of a non-negative potential V. If h is a smooth function in M, $\triangle_g + h$ is said to be coercive if there exists a positive constant C such that for any $u \in H^1(M)$,

$$\int_M \left(|\nabla u|_g^2 + h u^2 \right) dv_g \geqslant C \|u\|_{H^1(M)}^2$$

or, equivalently, if

$$\|u\|_{H_h^1} = \left(\int_M \left(|\nabla u|_g^2 + h u^2 \right) dv_g \right)^{\frac{1}{2}} \tag{1.8}$$

is an equivalent norm on $H^1(M)$. In this case, following Hebey, Pacard and Pollack [11], we define the constant S_h to be the smallest positive constant satisfying that for all $u \in H^1(M)$:

$$\|u\|_{L^{2^*}} \leq S_h^{\frac{1}{2^*}} \|u\|_{H_h^1}. \tag{1.9}$$

Our main result states the existence of a solution (φ, W) of (1.3)-(1.4) in the positive case under suitable smallness assumptions on the free data. It is stated as follows.

Theorem 1.1. *Let (M, g) be a closed compact Riemannian manifold of dimension $n \geq 3$ of positive Yamabe type such that g has no conformal Killing vector fields. Let V be a smooth nonnegative function on $\mathbb{R}$, $V \not\equiv 0$, and let ψ be a smooth function in M such that the operator $\triangle_g + \mathcal{R}_\psi$ is coercive. There exists a positive constant $\varepsilon(n, g, V, \psi)$ depending only on $n, g, \sup_{x \in M} V(\psi(x))$ and $S_{\mathcal{R}_\psi}$ as in (1.9) such that if the remaining part of the initial data (τ, π, U) satisfies*

$$\frac{n-1}{n} \tau^2(x) \leq 2V(\psi(x)) \quad \text{for all } x \in M, \tag{1.10}$$

the equality being strict somewhere in M, $\|\pi\|_\infty + \|U\|_\infty > 0$ and

$$\|\nabla \tau\|_\infty + \|\pi\|_\infty + \|U\|_\infty \leq \varepsilon(n, g, V, \psi), \tag{1.11}$$

then the conformal constraint system (1.3)-(1.4) has a solution (φ, W).

Remark 1.2. When the initial data satisfies condition (1.10), by the notations in (1.5), $\mathcal{B}_{\tau, \psi, V}$ is non-negative in M and positive somewhere. As we shall see in Section 2, when $\mathcal{B}_{\tau, \psi, V}$ is non-negative, then being of positive Yamabe type is a necessary condition.

To our knowledge, Theorem 1.1 is the first result in the non-CMC setting ($\nabla \tau \not\equiv 0$) when $\mathcal{B}_{\tau, \psi, V} \geq 0$. With this result we get the existence of admissible initial data in important cases such as the massive Klein-Gordon setting with nonzero potential or the positive cosmological constant case. Due to its importance we state the latter separately:

Corollary 1.3. *Let Λ be a positive constant. Then the vacuum conformal constraint system of equations with positive cosmological constant Λ, namely*

$$\begin{cases} \dfrac{4(n-1)}{n-2}\triangle_g\varphi + R(g)\varphi = \left(2\Lambda - \dfrac{n-1}{n}\tau^2\right)\varphi^{2^*-1} + |U + \mathcal{L}_g W|_g^2\,\varphi^{-2^*-1} \;, \\ \triangle_{g,conf}W = \dfrac{n-1}{n}\varphi^{2^*}\nabla\tau \;, \end{cases}$$

has a solution (φ, W) provided $U \not\equiv 0$ and

$$\|\tau\|_{C^1} + \|U\|_\infty \leq C(n, g, \Lambda) \;, \tag{1.12}$$

where $C(n, g, \Lambda)$ is some constant depending only on n, g and Λ.

In this scalar-field setting the smallness assumptions (1.10) and (1.11) only involve the scalar field ψ and the potential V, which is itself related to ψ by some wave equation that expresses the conservation of energy, see Wald [19]. This emphasizes the influence of ψ which appears to be the important parameter to consider.

There are several interesting results on systems like (1.3)-(1.4). They can be roughly classified according to two criteria: (i) the CMC (constant mean curvature) versus the non-CMC case, and, if we forget about the fact that $\mathcal{B}_{\tau,\psi,V}$ may change sign, (ii) the positive case, where $\mathcal{B}_{\tau,\psi,V} > 0$, versus the nonpositive case, where $\mathcal{B}_{\tau,\psi,V} \leq 0$. In the CMC setting ($\nabla\tau = 0$) the system (1.3)-(1.4) is semi-decoupled. Equation (1.4) is solvable, either assuming that there are no conformal Killing fields on M or assuming that $\pi\nabla\psi$ is orthogonal to such fields (which generically do not exist, see Beig-Chruściel-Schoen [3]). Its solution appears as a coefficient in (1.3). In the CMC-case when $\mathcal{B}_{\tau,\psi,V} \leq 0$, for instance in the vacuum case, the system is fully understood (see Isenberg [13] or Choquet-Bruhat, Isenberg and Pollack [6]). Partial results exist in the $\max_M \mathcal{B}_{\tau,\psi,V} > 0$ case, and we refer to Hebey, Pacard and Pollack [11], and Ngô and Xu [17]. In the non-CMC case, results were available only when $\mathcal{B}_{\tau,\psi,V} \leq 0$ and assuming smallness assumptions on the initial data. For near-CMC results see Allen-Clausen-Isenberg [1] or Dahl-Gicquaud-Humbert [8]. Results when U is small can be found in Holst-Nagy-Tsogtgerel [12] or Maxwell [16]. A few non-existence results exist for near-CMC initial data: see Isenberg-Ò Murchadha [14] or again Dahl-Gicquaud-Humbert [8]. A condition like our condition (1.12) is both a near-CMC assumption and a control on $\|U\|_\infty$, and Corollary 1.3 can be thought as a generalization of the available existence results for the vacuum conformal constraint system of equations to the more involved case where $\mathcal{B}_{\tau,\psi,V} > 0$.

The paper is organized as follows. In section 2 we comment on Theorem 1.1. We prove necessary conditions for the existence of solutions of (1.3)-(1.4) and show that the need of a control on the initial data is natural. Section 3 is devoted to the proof of the existence of a smallest solution for equation (1.3). Theorem 1.1 is proved in section 4 using a fixed-point argument.

2 Necessary conditions and non-existence results.

We discuss the assumptions of Theorem 1.1. Throughout this section we assume that V is a smooth nonnegative function in $\mathbb{R}$, not everywhere zero. We let

$$\mu_g = \inf_{\varphi \in H^1(M), \|\varphi\|_2 = 1} \int_M \left(|\nabla\varphi|^2 + c_n R(g)\varphi^2\right) dv_g \tag{2.1}$$

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be the coercivity constant of $\triangle_g + c_n R(g)$, where c_n is as in (1.5), and

$$\mu_{g,\psi} = \inf_{\varphi \in H^1(M), \|\varphi\|_2 = 1} \int_M \left(|\nabla \varphi|^2 + \mathcal{R}_\psi \varphi^2 \right) dv_g \tag{2.2}$$

be the analogue for $\triangle_g + \mathcal{R}_\psi$. The first result we prove shows that the coercivity of $\triangle_g + \mathcal{R}_\psi$ is a necessary condition to the existence of admissible initial data when $\mathcal{B}_{\tau,\psi,V}$ is non-negative. This shows, in some sense, the optimality of the assumptions required on ψ in Theorem 1.1. As a by-product we obtain an integral control on $|\nabla \psi|$, which in turns implies a strong geometric condition on the underlying manifold which shows a radically different behavior than in the vacuum case.

Proposition 2.1. *Let (ψ, τ, π, U) be an initial data set such that $\mathcal{B}_{\tau,\psi,V} \geq 0$ on M, where $\mathcal{B}_{\tau,\psi,V}$ is as in (1.5), but $\mathcal{B}_{\tau,\psi,V}$ is not everywhere zero. If a solution (φ, W) of (1.3)-(1.4) exists, then $\triangle_g + \mathcal{R}_\psi$ is coercive and there holds that*

$$\int_M |\nabla \psi|^2 dv_g < \int_M R(g) dv_g. \tag{2.3}$$

In particular, (M, g) is of positive Yamabe type and we get both that $\mu_{g,\psi} > 0$ and $\mu_g > 0$.

Proof. Using standard variational techniques and elliptic theory we easily obtain that there exists a smooth positive function $u_{g,\psi}$ with $\|u_{g,\psi}\|_2 = 1$ such that

$$\triangle_g u_{g,\psi} + \mathcal{R}_\psi u_{g,\psi} = \mu_{g,\psi} u_{g,\psi} \tag{2.4}$$

where $\mu_{g,\psi}$ is as in (2.2). Since (φ, W) solves (1.3)-(1.4) and $\mathcal{B}_{\tau,\psi,V}$ is non-negative, $\mathcal{B}_{\tau,\psi,V} \not\equiv 0$, integrating (1.3) against $u_{g,\psi}$ and using (2.4) shows that $\mu_{g,\psi} > 0$. It is well-known (see again [9]) that this implies the coercivity of $\triangle_g + \mathcal{R}_\psi$ which implies in particular that $\int_M \mathcal{R}_\psi dv_g > 0$ and yields (2.3). Assuming by contradiction that the Yamabe type of (M, g) is nonpositive, we get that there exists $\tilde{g} \in [g]$, where $[g]$ is the conformal class of g, with $R(\tilde{g}) \leq 0$ in M. Writing that $g = v^{4/(n-2)}\tilde{g}$ for $v > 0$, there holds that

$$\Delta_{\tilde{g}} v + c_n R(\tilde{g}) v = c_n R(g) v^{2^\star - 1}$$

Dividing the equation by $v^{2^\star - 1}$ and integrating the contradiction follows from (2.3). $\qquad \square$

Now we discuss a non-existence result which shows the necessity of a control on π depending on $\mathcal{B}_{\tau,\psi,V}$. More precisely, the following result by Hebey, Pacard and Pollack [11] holds true.

Proposition 2.2. *Let (τ, ψ) be smooth functions with $\mathcal{B}_{\tau,\psi,V} > 0$ in M, where $\mathcal{B}_{\tau,\psi,V}$ is as in (1.5). If π is a smooth function in M satisfying*

$$\int_M \pi^{\frac{n+2}{2n}} dv_g > \left(\frac{(n-1)^{n-1} c_n^{n-1}}{n^n} \right)^{\frac{n+2}{4n}} \frac{\int_M |R(g)|^{\frac{n+2}{4}} dv_g}{(\min_{x \in M} \mathcal{B}_{\tau,\psi,V}(x))^{\frac{(n-1)(n+2)}{4n}}},$$

then the system (1.3)-(1.4) admits no solutions with (ψ, τ, π, U) as initial data set for any smooth traceless and divergence-free $(2, 0)$-tensor U.

Proof. Let W be a smooth vector field in M. Following Hebey-Pacard-Pollack [11] we get that (1.3) has no solutions if

$$\int_M \mathcal{A}_{\pi,U}(W)^{\frac{n+2}{4n}} > \left(\frac{(n-1)^{n-1}}{n^n} \right)^{\frac{n+2}{4n}} \left(\min_M \mathcal{B}_{\tau,\psi,V} \right)^{-\frac{(n-1)(n+2)}{4n}} \int_M (\mathcal{R}_\psi^+)^{\frac{n+2}{4}} dv_g, \tag{2.5}$$

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where we used the notations of (1.5) and where for any function f we write $f^+ = \max(f, 0)$. We prove (2.5) by contradiction. We assume that (1.3) has a smooth positive solution φ. First, we integrate (1.3) to get

$$\int_M \mathcal{B}_{\tau,\psi,V} \varphi^{2^*-1} dv(g) + \int_M \mathcal{A}_{\pi,U}(W) \varphi^{-2^*-1} dv(g) = \int_M \mathcal{R}_\psi \varphi dv(g) \qquad (2.6)$$

and then we apply a Hölder inequality with parameters $\frac{n+2}{4}$ and $\frac{n+2}{n-2}$ to the right-hand side of (2.6). This yields

$$\int_M \mathcal{R}_\psi \varphi dv(g) \le \left(\int_M (\mathcal{R}_\psi^+)^{\frac{n+2}{4}} \mathcal{B}_{\tau,\psi,V}^{-\frac{n-2}{4}} dv(g) \right)^{\frac{4}{n+2}} \left(\int_M \mathcal{B}_{\tau,\psi,V} \varphi^{2^*-1} dv(g) \right)^{\frac{n-2}{n+2}} .$$

Independently, a Hölder inequality with parameters $\frac{4n}{n+2}$ and $\frac{4n}{3n-2}$ yields

$$\int_M \mathcal{A}_{\pi,U}(W)^{\frac{n+2}{4n}} \mathcal{B}_{\tau,\psi,V}^{\frac{3n-2}{4n}} dv(g) \le \left(\int_M \mathcal{A}_{\pi,U}(W) \varphi^{-2^*-1} dv(g) \right)^{\frac{n+2}{4n}} \left(\int_M \mathcal{B}_{\tau,\psi,V} \varphi^{2^*-1} dv(g) \right)^{\frac{3n-2}{4n}} ,$$

so that, letting $X = \left(\int_M \mathcal{B}_{\tau,\psi,V} \varphi^{2^*-1} dv(g) \right)^{\frac{4}{n+2}}$, equation (2.6) gives

$$\left(\int_M (\mathcal{R}_\psi^+)^{\frac{n+2}{4}} \mathcal{B}_{\tau,\psi,V}^{-\frac{n-2}{4}} dv(g) \right)^{\frac{4}{n+2}} \geqslant X + \left(\int_M \mathcal{A}_{\pi,U}(W)^{\frac{n+2}{4n}} \mathcal{B}_{\tau,\psi,V}^{\frac{3n-2}{4n}} dv(g) \right)^{\frac{4n}{n+2}} X^{1-n} . \qquad (2.7)$$

Let K_X be the right hand side in (2.7). The minimum value of K_X as a function of X is:

$$\min_{X>0} K_X = \frac{n}{(n-1)^{\frac{n-1}{n}}} \left(\int_M \mathcal{A}_{\pi,U}(W)^{\frac{n+2}{4}} \mathcal{B}_{\tau,\psi,V}^{\frac{3n-2}{4n}} dv_g \right)^{\frac{4}{n+2}} \qquad (2.8)$$

and the non-existence condition (2.5) easily follows from (2.7) and (2.8) by contradiction. Then Proposition 2.2 follows from (2.5) since $\mathcal{A}_{\pi,U}(W) \ge c_n \pi^2$ and $\mathcal{R}_\psi^+ \le c_n |R(g)|$ by (1.5). □

3 A minimal solution of the Einstein-Lichnerowicz equation

In the constant mean curvature setting the constraint system is completely decoupled and it reduces to the Einstein-Lichnerowicz equation (1.3). We now investigate (1.3) independently and for the sake of clarity consider the following equation:

$$\triangle_g u + hu = fu^{2^*-1} + \frac{a}{u^{2^*+1}} , \qquad (EL_a)$$

where h, f and a are smooth functions on M. In the following we assume that $\triangle_g + h$ is coercive, $\max_M f > 0$ and a is nonnegative and nonzero. Using repeatedly the sub and super solution method we prove that each time equation (EL_a) has a smooth positive solution then it has a smallest solution for the L^∞-norm:

Proposition 3.1. *Let $a \ge 0$ be a nonzero smooth function in M. Assume that $\triangle_g + h$ is coercive and $\max_M f > 0$. If (EL_a) has a smooth positive solution then there exists a smooth positive function $\varphi(a)$ solving (EL_a) such that for any other solution φ, with $\varphi \not\equiv \varphi(a)$, there holds that $\varphi(a) < \varphi$ in M. Moreover, $\varphi(a)$ is stable in the sense that for any $\theta \in H^1(M)$,*

$$\int_M \left(|\nabla \theta|^2 + \left[h - (2^*-1) f \varphi(a)^{2^*-2} + (2^*+1) \frac{a}{\varphi(a)^{2^*+2}} \right] \theta^2 \right) dv_g \ge 0 ,$$

and $a \to \varphi(a)$ is also nondecreasing with respect to a in the sense that if $a_1 \le a_2$ in M, provided that $\varphi(a_1)$ and $\varphi(a_2)$ exist, then there holds that $\varphi(a_1) \le \varphi(a_2)$.

Remark 3.2. We prove Proposition 3.1 assuming that a is smooth but the result still holds if a is only continuous. In this case the minimal solution we obtain belongs to $C^{1,\alpha}(M)$ for any $0 < \alpha < 1$.

Proof. Let $a \geq 0$ be a nonzero smooth function such that (EL_a) has a solution. We start proving that there exists a positive number that bounds from below all the solutions of (EL_a). First, let us notice that there always exist sub-solutions of (EL_a) as small as we want. Indeed, for any $\delta \geq 0$ we let u_δ be the unique solution of

$$\triangle_g u_\delta + h u_\delta = a - \delta f^- - \delta \tag{3.1}$$

where $f^- = -\min(f, 0)$. Since a is nonnegative and nonzero, the maximum principle shows that $u_0 > 0$ in M. Since

$$(\triangle_g + h)(u_0 - u_\delta) = \delta f^- + \delta \ ,$$

we obtain by standard elliptic theory that $\|u_\delta - u_0\|_\infty \to 0$ as δ tends to 0. Let $\delta_0 > 0$ be small enough in order to have $u_{\delta_0} > 0$. Then for ε small enough,

$$v_\varepsilon = \varepsilon u_{\delta_0} \tag{3.2}$$

is a strict sub-solution of (EL_a) since, by (3.1),

$$\triangle_g v_\varepsilon + h v_\varepsilon = \varepsilon a - \varepsilon \delta_0 f^- - \varepsilon \delta_0 < \frac{a}{v_\varepsilon^{2^*+1}} + f v_\varepsilon^{2^*-1}.$$

Now we claim that there exists some $\varepsilon_0 > 0$ such that for any positive solution φ of (EL_a) there holds

$$\varphi > v_{\varepsilon_0} \tag{3.3}$$

in M, where v_{ε_0} is as in (3.2). We prove the claim by contradiction and assume that there exists φ_ε solution of (EL_a), and $x_\varepsilon \in M$, such that $\varphi(x_\varepsilon) \leq v_\varepsilon(x_\varepsilon)$ for all $\varepsilon > 0$. Then, for some $\tilde{\varepsilon} \in (0, \varepsilon)$, and some $\tilde{x}_\varepsilon \in M$,

$$1 = \inf_M \frac{\varphi}{v_{\tilde{\varepsilon}}} = \frac{\varphi(\tilde{x}_\varepsilon)}{v_{\tilde{\varepsilon}}(\tilde{x}_\varepsilon)} \ .$$

In particular, we obtain that

$$v_{\tilde{\varepsilon}}(\tilde{x}_\varepsilon) = \varphi(\tilde{x}_\varepsilon) \ \text{ and } \ \triangle_g \varphi(\tilde{x}_\varepsilon) \leq \triangle_g v_{\tilde{\varepsilon}}(\tilde{x}_\varepsilon)$$

which is impossible since $v_{\tilde{\varepsilon}}$ is a strict subsolution of (EL_a).

We prove now the existence of a minimal solution of (EL_a). We follow here the arguments in Sattinger [18]. For $x \in M$ and $u > 0$ we let

$$F(x, u) = f(x)u(x)^{2^*-1} + \frac{a(x)}{u(x)^{2^*+1}} - h(x)u(x).$$

Let ψ be a solution of (EL_a) and let w be a strict subsolution of (EL_a) which is less than any positive solution of (EL_a). We proved the existence of such a w in (3.3). Also we let $K > 0$ be large enough such that for any $x \in M$, and any $\min_M w \leq u \leq \max_M \psi$,

$$F(x, u) + Ku \geq 0 \ \text{ and } \ \frac{\partial F}{\partial u}(x, u) + K \geq 0. \tag{3.4}$$

For any smooth positive function u, we define Tu as the unique solution of

$$\triangle_g Tu + KTu = F(\cdot, u) + Ku. \tag{3.5}$$

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As a first remark, for any two positive functions u and v in the range

$$\min_M w \leq u, v \leq \max_M \psi$$

we have:

$$\big(\triangle_g + K\big)(Tu - Tv)(x) = F(x, u) - F(x, v) + K\big(u(x) - v(x)\big).$$

Then, by the strong maximum principle, we obtain that

$$Tu < Tv \text{ as long as } u \leq v \text{ and } u \not\equiv v. \tag{3.6}$$

The iterative sub and super solution method applied in the range $w \leq \varphi \leq \psi$ and starting from the strict sub-solution w provides a sequence $v_n = T^n w$ which is non decreasing by the maximum principle and converges to a fixed-point of T, that is to say a solution of (EL_a) (see [18] for more details). We shall call this solution $\varphi(a)$:

$$\varphi(a) = \lim_{n \to \infty} T^n w. \tag{3.7}$$

By standard elliptic theory, $\varphi(a)$ is smooth. Note in passing that all the above arguments still work if we only assume that a is continuous, but in this case $\varphi(a)$ constructed as in (3.7) will only be of class $C^{1,\alpha}$ for any $0 < \alpha < 1$.

Now we show that $\varphi(a)$ does not depend on ψ and on w. First, $\varphi(a)$ as in (3.7) does not depend on ψ. We let ψ_1 and ψ_2 be two solutions of (EL_a). We let K_i, $i = 1, 2$ be positive constants satisfying (3.4) in $[\min_M w; \max_M \psi_i]$, T_i be the operator defined as in (3.5) and φ_i the associated solution as in (3.7). Since $(T_1^n w)$ is non decreasing there holds $\varphi_1 \geq w$. If we assume for instance that $\max_M \psi_1 \leq \max_M \psi_2$ then $\varphi_1 \in [\min_M w; \max_M \psi_2]$ and thus, by (3.7) and the maximum principle there holds $\varphi_2 \leq \varphi_1$ since $T_2(\varphi_1) = \varphi_1$. But then φ_2 is a solution of (EL_a) with $\min_M w \leq \varphi_2 \leq \max_M \psi_1$ and thus, once again by the maximum principle, $\varphi_1 \leq \varphi_2$. This proves that $\varphi(a)$ does not depend on ψ. Now we prove that $\varphi(a)$ does not depend on the strict subsolution w, provided that w is less than any positive solution of (EL_a). Indeed, for any ψ solution of (EL_a), if w_1 and w_2 are two such subsolutions, and φ_1 and φ_2 are the associated solutions as in (3.7), then there holds $w_1 \leq \varphi_2$ and $w_2 \leq \varphi_1$. We conclude once again with the maximum principle that $\varphi_1 \leq \varphi_2$ and $\varphi_2 \leq \varphi_1$.

By the definition of $\varphi(a)$ in (3.7), and what we just proved, for any ψ solution of (EL_a) there holds that $w < \varphi(a) \leq \psi$ where w is a subsolution that is less than any solution of (EL_a). With (3.6) we obtain the desired property:

$$\varphi(a) < \psi \text{ or } \varphi(a) \equiv \psi. \tag{3.8}$$

The stability of $\varphi(a)$ is a consequence of the minimality of $\varphi(a)$. We denote by λ_0 the first eigenvalue of the linearized operator of equation (EL_a) at $\varphi(a)$. The stability of $\varphi(a)$ as stated in Proposition 3.1 amounts to say that $\lambda_0 \geq 0$. Assume by contradiction that $\lambda_0 < 0$ and denote by ψ_0 the associated positive eigenvector. Let w be a subsolution that is less than any solution of (EL_a). Let $\varphi_\delta = \varphi(a) - \delta\psi_0$ for any positive δ. For $\delta > 0$ small enough one has

$$w < \varphi_\delta < \varphi(a)$$

and a straightforward calculation shows that

$$\triangle_g \varphi_\delta + h\varphi_\delta - f\varphi_\delta^{2^*-1} - \frac{a}{\varphi_\delta^{2^*+1}} = -\delta\lambda_0\psi_0 + o(\delta) > 0$$

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so that φ_δ is a strict supersolution of (EL_a) satisfying $w < \varphi_\delta < \varphi(a)$ for δ small enough. By the iterative sub and super solution method we then get a solution ψ of (EL_a) such that $w < \psi < \varphi_\delta$, and this is in contradiction with (3.8).

Finally, if $a_1 \le a_2$ are nonnegative nonzero functions on M, $\varphi(a_2)$ is a super solution of equation (EL_a) with $a = a_1$. By the minimality of $\varphi(a_1)$ we then have $\varphi(a_1) \le \varphi(a_2)$. □

4 The fixed-point method: proof of Theorem 1.1

In this section we prove Theorem 1.1 using the results obtained in the previous section. Before we start let us recall some basic elliptic properties of the operator $\triangle_{g,conf}$ that can be found for instance in Isenberg and Ò Murchadha [14]. The following proposition can be found in [14].

Proposition 4.1. *Let (M, g) be a closed compact Riemannian manifold of dimension $n \ge 3$ such that g has no conformal Killing fields. Let X be a smooth vector field in M. Then there exists a unique solution W of*

$$\triangle_{g,conf} W = X.$$

Also, there exists a constant $C_0 > 0$ that depends only on n and g such that

$$\|W\|_{C^{1,\alpha}} \le C_0 \|X\|_\infty$$

for some positive α. As a straightforward consequence, there exists a constant C_1 still depending only on n and g such that $\|\mathcal{L}_g W\|_\infty \le C_1 \|X\|_\infty$.

The proof of Theorem 1.1 is obtained through a standard fixed-point argument. We develop the proof in what follows.

Obtaining a first estimate. Let (M, g) be a closed compact Riemannian manifold of dimension $n \ge 3$ of positive Yamabe type such that g has no conformal Killing vector fields. Let V be a smooth nonnegative function in $\mathbb{R}$, non everywhere zero and ψ be a smooth function in M such that $\triangle_g + \mathcal{R}_\psi$ is coercive. Also let π and U be such that (π, U) is not everywhere zero in M, and let τ be a smooth function in M such that

$$\frac{n-1}{n} \tau^2(x) \le 2V(\psi(x)) \text{ for all } x \in M,$$

the equality being strict somewhere. This means, with the notations of (1.5), that $\mathcal{B}_{\tau,\psi,V}$ is nonnegative in M and positive somewhere. In order to define the mapping to which we are going to apply Schauder's fixed-point Theorem, we need to get some important preliminary estimate based on (1.11). Let W be a smooth vector field in M. We let

$$C(n, g, V, \psi) = C(n) V_g^{-1} \left(2c_n S_{\mathcal{R}_\psi} \max_{x \in M} V(\psi(x)) \right)^{1-n} \left(\int_M \mathcal{R}_\psi dv_g \right)^{-\frac{2^*}{2}} \tag{4.1}$$

where V_g is the volume of (M, g), $\mathcal{R}_\psi$ is as in (1.5), $S_{\mathcal{R}_\psi}$ is as in (1.9) and $C(n) = \frac{1}{n-2} (2(n-1))^{-\frac{2^*}{2}}$. By (1.9) the constant $S_{\mathcal{R}_\psi}$ only depends on n, g and on the coercivity constant of $\triangle_g + \mathcal{R}_\psi$, hence on n, g and $\nabla \psi$. We consider the equation

$$\triangle_g \varphi + \mathcal{R}_\psi \varphi = \mathcal{B}_{\tau,\psi,V} \varphi^{2^*-1} + C(n, g, V, \psi) \varphi^{-2^*-1} \tag{4.2}$$

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By the result in Hebey-Pacard-Pollack [11], and since by (1.5) we have that $2c_n V \geq \mathcal{B}_{\tau,\psi,V}$, (4.2) has a smooth positive solution. Since we assumed $\triangle_g + \mathcal{R}_\psi$ coercive, using Proposition 3.1 we can let φ_m be the minimal solution of (4.2) and let

$$N_m = \|\varphi_m\|_\infty. \tag{4.3}$$

Let $L_+^\infty(M)$ be the set of non negative bounded functions in M. Regarding the vector equation (1.4), since we have assumed that g has no Killing vector fields, for any $\eta \in L_+^\infty(M)$ we can use Proposition 4.1 to let $W(\eta)$ be the unique vector field solution of

$$\triangle_{g,conf} W(\eta) = \frac{n-1}{n} \eta^{2^*} \nabla\tau - \pi\nabla\psi. \tag{4.4}$$

Proposition 4.1 shows that

$$\|\mathcal{L}_g W(\eta)\|_\infty \leq C_1 \left(\|\nabla\tau\|_\infty \|\eta\|_\infty^{2^*} + \|\pi\|_\infty \|\nabla\psi\|_\infty \right). \tag{4.5}$$

By (4.1), (4.2) and (4.3), N_m depends only on $n, g, S_{\mathcal{R}_\psi}$ and $\max_M V(\psi)$. Using (1.5) and (4.5) it is easily seen that there exists a positive constant $\varepsilon(n, g, V, \psi)$ depending only on $n, g, S_{\mathcal{R}_\psi}$, and $\max_M V(\psi)$, such that whenever

$$\|\nabla\tau\|_\infty + \|\pi\|_\infty + \|U\|_\infty \leq \varepsilon(n, g, V, \psi) , \tag{4.6}$$

then

$$\|\mathcal{A}_{\pi,U}(W(\eta))\|_\infty < C(n, g, V, \psi) \tag{4.7}$$

for any $\|\eta\|_\infty \leq N_m$, where $\mathcal{A}_{\pi,U}(W(\eta))$ is as in (1.5). After a straightforward computation, using (1.5) and (4.5), one sees that, in order to obtain (4.7), it is enough to assume that

$$\varepsilon(n, g, V, \psi)^2 < \frac{C(n, g, V, \psi)}{(3 + 4C_1^2(N_m^{2 \cdot 2^*} + \|\nabla\psi\|_\infty^2)) c_n} \tag{4.8}$$

where c_n is as in (1.5), C_1 is obtained in Proposition 4.1, $C(n, g, V, \psi)$ is as in (4.1) and N_m is as in (4.3). Now we construct the map $\mathcal{T}$ to which we are going to apply Schauder's fixed point theorem.

Definition of the mapping $\mathcal{T}$. From now on, we will always assume that (4.6) is satisfied, so that (4.7) holds true. For any positive N, we define

$$B_N = \{\eta \in L_+^\infty(M), \|\eta\|_\infty \leq N\}. \tag{4.9}$$

An easy claim is that for any vector field W of class C^1 in M, $\mathcal{A}_{\pi,U}(W)$ as in (1.5) is continuous, non-negative and positive somewhere in M. Obviously, $\mathcal{A}_{\pi,U}(W)$ is continuous and non-negative, and we just need to prove that it is positive somewhere. By (1.5) this is automatically true if π is not everywhere zero. In case $\pi \equiv 0$ there might be that there exists a C^1 vector field in M such that $U + \mathcal{L}_g W = 0$ everywhere. Taking the divergence of this equality in the weak sense yields, since U is divergence-free:

$$\triangle_{g,conf} W = 0$$

in the weak sense. Since g has no Killing fields this implies $W = 0$ and hence $U = 0$, which is impossible since we assumed (π, U) non everywhere zero. Now by (1.5) it is easily seen that

$$C(n, g, V, \psi) \leq C(n) V_g^{-1} \left(S_{\mathcal{R}_\psi} \max_{x \in M} \mathcal{B}_{\tau,\psi,V}(x) \right)^{1-n} \left(\int_M \mathcal{R}_\psi dv_g \right)^{-\frac{2^*}{2}} \tag{4.10}$$

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where we used the notations of (1.5) and where $C(n, g, V, \psi)$ is as in (4.1). Now we consider the equation

$$\triangle_g \varphi + \mathcal{R}_\psi \varphi = \mathcal{B}_{\tau,\psi,V} \varphi^{2^\star - 1} + \mathcal{A}_{\pi,U} \left(W(\eta) \right) \varphi^{-2^\star - 1} . \tag{4.11}$$

We claim that (4.11) has a smooth positive solution for any $\eta \in B_{N_m}$. We just proved that $\mathcal{A}_{\pi,U}(W(\eta))$ is never zero. Thus we can construct subsolutions of (4.11) as small as we want as we did in Section 3, see (3.2). On the other hand, by (4.7) and (4.10), there holds that for any $\eta \in B_{N_m}$ and any $\delta > 0$ sufficiently small,

$$\mathcal{A}_{\pi,U}(W(\eta)) + \delta \le C(n) V_g^{-1} \left(S_{\mathcal{R}_\psi} \max_{x \in M} \mathcal{B}_{\tau,\psi,V}(x) \right)^{1-n} \left(\int_M \mathcal{R}_\psi dv_g \right)^{-\frac{2^\star}{2}} .$$

Then the existence result by Hebey-Pacard-Pollack, namely Theorem 3.1 and equation (3.3) in [11], applies to (4.11) when replacing $\mathcal{A}_{\pi,U}(W(\eta))$ by $\mathcal{A}_{\pi,U}(W(\eta)) + \delta$ and provides us with a strict super solution of (4.11). Since $\mathcal{A}_{\pi,U}(W(\eta))$ is nonzero for all $\eta \in B_{N_m}$, and since it is smooth, Proposition 3.1 shows that (4.11) possesses a minimal smooth positive solution $\varphi(\mathcal{A}_{\pi,U}(W(\eta)))$, where we use the same notations as in Proposition 3.1. The following map:

$$\mathcal{T} : \eta \mapsto \mathcal{T}(\eta) = \varphi \left(\mathcal{A}_{\pi,U}(W(\eta)) \right) \tag{4.12}$$

is thus well-defined in B_{N_m}. It is clear that a fixed point of $\mathcal{T}$ is a solution of the constraint system. As a consequence of the monotonicity property of the minimal solution in Proposition 3.1 along with (4.7) and the very definition of N_m in (4.3) we obtain that, for any $\eta \in B_{N_m}$,

$$0 < \mathcal{T}(\eta) \le N_m. \tag{4.13}$$

Hence B_{N_m} is stable under $\mathcal{T}$ and $\mathcal{T}$ maps B_{N_m} into itself. Now we prove that $\mathcal{T}$ is continuous in B_{N_m}.

$\mathcal{T}$ **is continuous in** B_{N_m}**.** First we claim that there exists a positive real number δ_0 such that for any $\eta \in B_{N_m}$, and any $x \in M$,

$$\mathcal{T}(\eta)(x) \ge \delta_0. \tag{4.14}$$

To prove this claim we pick a sequence $(\eta_k)_k$ in B_{N_m} and show that there holds

$$\liminf_{k \to +\infty} \min_{x \in M} \mathcal{T}(\eta_k)(x) > 0. \tag{4.15}$$

We consider the associated $W(\eta_k)$ as in (4.4). By Proposition 4.1, if we choose $\varepsilon(n, g, V, \psi)$ in (4.6) such that

$$\varepsilon(n, g, V, \psi) < \frac{N_m^{2^\star}}{\frac{n-1}{n} N_m^{2^\star} + \|\nabla \psi\|_\infty} ,$$

then there holds

$$\|W(\eta_k)\|_{C^{1,\alpha}} \le C_0 N_m^{2^\star} \tag{4.16}$$

so that, up to a subsequence, we can assume that $W(\eta_k)$ converges to some W_0 in the $C^{1,\alpha}(M)$-topology for some $0 < \alpha < 1$. As noticed in a previous remark right after defining $\mathcal{T}$, $\mathcal{A}_{\pi,U}(W_0)$ is non-negative and positive somewhere in M. We denote by x_0 its maximum point and choose $0 < r < i_g(M)$ so as to have

$$\mathcal{A}_{\pi,U}(W_0)(x) \ge \frac{1}{2} \mathcal{A}_{\pi,U}(W_0)(x_0)$$

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for all $x \in B_{x_0}(2r)$. Let λ be a smooth nonnegative function, compactly supported in $B_{x_0}(2r)$ and equal to 1 in $B_{x_0}(r)$. Since $W(\eta_k) \xrightarrow{C^{1,\alpha}} W_0$ as k goes to infinity one has, for k large enough, in M, that

$$\mathcal{A}_{\pi,U}(W(\eta_k)) \geq \frac{1}{2}\lambda \mathcal{A}_{\pi,U}(W_0).$$

The monotonicity property in Proposition 3.1 and the definition of $\mathcal{T}$ in (4.12) thus show that

$$\mathcal{T}(\eta_k) \geq \varphi\left(\frac{1}{2}\lambda \mathcal{A}_{\pi,U}(W_0)\right) ,$$

where the right-hand side is a smooth positive function in M, which shows (4.15).

Now we prove the continuity of $\mathcal{T}$. We let $\eta_k \in B_{N_m}$ be a sequence of nonnegative functions in M converging uniformly to some $\eta_0 \in B_{N_m}$. There holds that

$$\triangle_g \mathcal{T}(\eta_k) + \mathcal{R}_\psi \mathcal{T}(\eta_k) = \mathcal{B}_{\tau,\psi,V} \mathcal{T}(\eta_k)^{2^*-1} + \mathcal{A}_{\pi,U}(W(\eta_k))\mathcal{T}(\eta_k)^{-2^*-1}$$

for all k. By (4.13) and (4.14), standard elliptic theory shows that $\mathcal{T}(\eta_k)$ converges in the $C^2(M)$-topology, up to a subsequence, to some $\mathcal{T}_0$ solution of

$$\triangle_g \mathcal{T}_0 + \mathcal{R}_\psi \mathcal{T}_0 = \mathcal{B}_{\tau,\psi,V} \mathcal{T}_0^{2^*-1} + \mathcal{A}_{\pi,U}(W(\eta_0))\mathcal{T}_0^{-2^*-1}. \qquad (4.17)$$

There holds then by Proposition 3.1: either $\mathcal{T}_0 = \mathcal{T}(\eta_0)$ or $\mathcal{T}_0 > \mathcal{T}(\eta_0)$ everywhere. We proceed by contradiction and assume that $\mathcal{T}_0 > \mathcal{T}(\eta_0)$. We then define for any $t \in [0; 1]$

$$m(t) = I_0\big(t\mathcal{T}(\eta_0) + (1-t)\mathcal{T}_0\big) ,$$

where I_0 is defined for any positive $\eta \in H^1(M)$ and is the energy associated to (4.17):

$$I_0(\eta) = \frac{1}{2}\int_M \left(|\nabla\eta|^2 + \mathcal{R}_\psi\eta^2\right) dv_g - \frac{1}{2^*}\int_M \mathcal{B}_{\tau,\psi,V}\eta^{2^*} dv_g + \frac{1}{2^*}\int_M \mathcal{A}_{\pi,U}(W(\eta_0))\eta^{-2^*} dv_g. \quad (4.18)$$

By proposition 3.1 each $\mathcal{T}(\eta_k)$ is a stable solution, and thus $\mathcal{T}_0$ is stable. Hence $m''(0) \geq 0$. Using (4.18) we can compute $m^{(3)}(t)$ for any $t \in [0,1]$, where $m^{(3)}$ is the third derivative of m. There holds

$$m^{(3)}(t) = -(2^*-1)(2^*-2)\int_M \mathcal{B}_{\tau,\psi,V}\big(t\mathcal{T}(\eta_0) + (1-t)\mathcal{T}_0\big)^{2^*-3}(\mathcal{T}(\eta_0) - \mathcal{T}_0)^3 dv_g$$

$$-(2^*+1)(2^*+2)\int_M \mathcal{A}_{\pi,U}(W(\eta_0))\big(t\mathcal{T}(\eta_0) + (1-t)\mathcal{T}_0\big)^{-2^*-3}(\mathcal{T}(\eta_0) - \mathcal{T}_0)^3 dv_g.$$

Since $\mathcal{T}_0 > \mathcal{T}(\eta_0)$ and $\mathcal{B}_{\tau,\psi,V}$ is nonnegative nonzero, $m^{(3)}(t)$ is positive for all $t \in (0,1)$. Hence m'' is a positive function of t for $0 < t \leq 1$ and m' is increasing in $(0,1)$. But this is impossible since both $\mathcal{T}_0$ and $\mathcal{T}(\eta_0)$ are solutions of (4.17) and there thus holds $m'(0) = m'(1) = 0$. Hence $\mathcal{T}_0 = \mathcal{T}(\eta_0)$ and in particular $\mathcal{T}$ is continuous. In order to apply the Schauder's fixed point theorem, it remains to prove the precompactness of $\mathcal{T}(B_{N_m})$, which itself follows from the compactness of $\mathcal{T}$.

Compactness of $\mathcal{T}$ and conclusion. Clearly, B_{N_m} is a closed convex set in $L_+^\infty(M)$. It remains to show that $\mathcal{T}(B_{N_m})$ is compact to conclude. By (4.7), (4.13) and (4.14), for any $\eta \in B_{N_m}$, $\mathcal{T}(\eta)$ satisfies $\delta_0 \leq \mathcal{T}(\eta) \leq N_m$ and:

$$\triangle_g \mathcal{T}(\eta) + \mathcal{R}_\psi \mathcal{T}(\eta) \leq \|\mathcal{B}_{\tau,\psi,V}\|_\infty N_m^{2^*-1} + \delta_0^{-2^*-1} C(n,g,V,\psi) ,$$

where $C(n, g, V, \psi)$ is as in (4.1). By standard elliptic theory $\mathcal{T}(B_{N_m})$ is thus bounded in $C^1(M)$. By the compactness of the embedding $C^1(M) \subset L^\infty(M)$ we then get that $\mathcal{T}(B_{N_m})$ is a compact set of $L^\infty_+(M)$. Applying Schauder's fixed point theorem yields the existence of a fixed-point of $\mathcal{T}$ on B_{N_m}, i.e. a solution of the constraint system, and concludes the proof of Theorem 1.1.

References

[1] P. T. Allen, A. Clausen, and J. Isenberg. Near-constant mean curvature solutions of the Einstein constraint equations with non-negative Yamabe metrics. *Classical Quantum Gravity*, 25(7):075009, 15, 2008.

[2] R. Bartnik and J. Isenberg. The constraint equations. In *The Einstein equations and the large scale behavior of gravitational fields*, pages 1–38. Birkhäuser, Basel, 2004.

[3] R. Beig, P. T. Chruściel, and R. Schoen. KIDs are non-generic. *Ann. Henri Poincaré*, 6(1):155–194, 2005.

[4] Y. Choquet-Bruhat and R. Geroch. Global aspects of the Cauchy problem in general relativity. *Comm. Math. Phys.*, 14(4):329–335, 1969.

[5] Y. Choquet-Bruhat, J. Isenberg, and D. Pollack. Applications of theorems of jean leray to the Einstein-scalar field equations. *J. Fixed Point Theory Appl.*, 1(1):31–46, 2007.

[6] Y. Choquet-Bruhat, J. Isenberg, and D. Pollack. The constraint equations for the Einstein-scalar field system on compact manifolds. *Classical Quantum Gravity*, 24(4):809–828, 2007.

[7] P. T. Chruściel, G. J. Galloway, and D. Pollack. Mathematical general relativity: a sampler. *Bull. Amer. Math. Soc. (N.S.)*, 47(4):567–638, 2010.

[8] M. Dahl, R. Gicquaud, and E. Humbert. A limit equation associated to the solvability of the vacuum Einstein constraint equations using the conformal method. *Duke Math. J.* To appear.

[9] O. Druet, E. Hebey, and F. Robert. *Blow-up theory for elliptic PDEs in Riemannian geometry*, volume 45 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2004.

[10] Y. Fourès-Bruhat. Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta Math.*, 88:141–225, 1952.

[11] E. Hebey, F. Pacard, and D. Pollack. A variational analysis of Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds. *Comm. Math. Phys.*, 278(1):117–132, 2008.

[12] M. Holst, G. Nagy, and G. Tsogtgerel. Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions. *Comm. Math. Phys.*, 288(2):547–613, 2009.

[13] J. Isenberg. Constant mean curvature solutions of the Einstein constraint equations on closed manifolds. *Classical Quantum Gravity*, 12(9):2249–2274, 1995.

[14] J. Isenberg and N. Ó Murchadha. Non-CMC conformal data sets which do not produce solutions of the Einstein constraint equations. *Classical Quantum Gravity*, 21(3):S233–S241, 2004. A spacetime safari: essays in honour of Vincent Moncrief.

[15] A. Lichnerowicz. L'intégration des équations de la gravitation relativiste et le problème des n corps. *J. Math. Pures Appl. (9)*, 23:37–63, 1944.

[16] D. Maxwell. A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature. *Math. Res. Lett.*, 16(4):627–645, 2009.

[17] Q. A. Ngô and X. Xu. Existence results for the Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds. *Adv. Math.*, 230(4-6):2378–2415, 2012.

[18] D. H. Sattinger. Monotone methods in nonlinear elliptic and parabolic boundary value problems. *Indiana Univ. Math. J.*, 21:979–1000, 1971/72.

[19] R. M. Wald. *General relativity.* University of Chicago Press, Chicago, IL, 1984.

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